Rocket Dynamics - Space Travel with Rockets

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1 Introduction

This document analyses some general features of rockets and their consequences for space travel. Two kind of rockets are considered: with direct propulsion (fuel and propulsion substance are the same) and indirect propulsion (fuel and propulsion substance are two different substances). Furthermore both types of rockets are ideal. That is, each has the folowing perfect features:

- it has only one stage, ie. it doesn't eject anything else but exhaust gases;
- its engine is 100% efficient, ie. all the potential energy contained in its fuel is converted fully to kinetic enery of the rocket and exhaust gases;
- the exhaust gasses (= ejected propulsion sunstance) are expelled purely backwards, ie. the velocity of all particals in the exhaust flame is pure longitudinal, there is no transversal component.

A illustration of a direct propulsion rocket is given in figure (1).

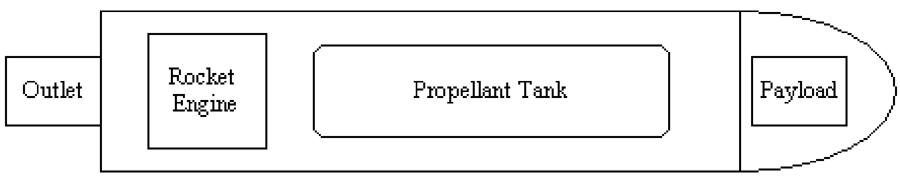

Figure 1: Direct propulsion rocket

The analisys leads to the conclusion that interstellar space travel is practically impossible. Even with rockets powered by nuclear fusion. We will never be able to reach the stars ...

2 Direct Propulsion Dynamics

The intial mass of the propulsion substance of the rocket is named m_p, the mass of the empty (ie. empty of propulsion substance) rocket is m_e. The engine burns for a period T at constant power P, producing a constant mass flow Φ with exhaust speed V (ie. the velocity at which the exhaust gases are expelled from the engine outlet). The total mass m of the rocket at time t then is then

$$m = \begin{cases} m_e + m_p - \Phi t & 0 \leq t \leq T \\ m_e & T < t \end{cases} \tag{1}$$

We will now use the law of conservation of momentum to deduce the motion and force equations for our rocket. At time t the momentum of our rocket is

$$p = mv \tag{2}$$

At time $t + \mathrm{d}t$ an amount $\mathrm{d}m$ of gas has been expelled backwards. But the momentum is conserved, thus

$$p = (m - \mathrm{d}m)(v + \mathrm{d}v) + \mathrm{d}m(v - V) \tag{3}$$

Combining equations (2) and (3) and ignoring second order infinitesimals leads to

$$m\mathrm{d}v - \mathrm{d}mV = 0 \tag{4}$$

Deviding by $\mathrm{d}t$ and realising that $\Phi = \mathrm{d}m/\mathrm{d}t$ and $a = \mathrm{d}v/\mathrm{d}t$ leads to

$$ma - \Phi V = 0 \tag{5}$$

$$\Leftrightarrow ma = \Phi V \tag{6}$$

$$\Leftrightarrow F = \Phi V \tag{7}$$

Where F is the force acting on the rocket at time t. As expected F is a constant.

Next we will calculate the velocity of the rocket. Using equations (6) and (1) we get

$$v = \int_0^t \mathrm{d}t \frac{\Phi V}{m} \tag{8}$$

$$= \int_0^t \mathrm{d}t \frac{\Phi V}{m_e + m_p - \Phi t} \tag{9}$$

$$= -V \ln(m_e + m_p - \Phi t)\Big|_0^t \tag{10}$$

$$= V \ln\left(\frac{m_e + m_p}{m_e + m_p - \Phi t}\right) \tag{11}$$

The final, or end, velocity v_{end} of the rocket thus is (using $m_p = \Phi T$ from (1))

$$v_{end} = v(T) \tag{12}$$

$$= V \ln\left(1 + \frac{m_p}{m_e}\right) \tag{13}$$

3 Energy Considerations

The rocket's engine produces an amount of energy dE which is fully converted into kinetic energy of the rocket and exhaust gases. At time $t + \mathrm{d}t$ the kinetic energy E of the rocket and the during period dt expelled gases is (again ignoring second order infinitesimals)

$$E(t+\mathrm{d}t) = \frac{1}{2}(m-\mathrm{d}m)(v+\mathrm{d}v)^2 + \mathrm{d}m(v-V)^2 \tag{14}$$

$$= \frac{1}{2}mv^2 + mv\mathrm{d}v - \mathrm{d}m\,vV + \frac{1}{2}\mathrm{d}mV^2 \tag{15}$$

The energy of the same components involved at time t was

$$E(t) = \frac{1}{2}mv^2 \tag{16}$$

Thus

$$\mathrm{d}E = mv\mathrm{d}v - \mathrm{d}m\,vV + \frac{1}{2}\mathrm{d}mV^2 \tag{17}$$

Dividing this by dt and then substituting (6) leads to

$$P = \frac{\mathrm{d}E}{\mathrm{d}t} \tag{18}$$

$$= mva - \Phi vV + \frac{1}{2}\Phi V^2 \tag{19}$$

$$= \frac{1}{2}\Phi V^2 \tag{20}$$

As expected, P is a constant!

The total amount of energy produced by the engine is of course

$$\Delta E = \int_0^T \mathrm{d}t\,P \tag{21}$$

$$= \frac{1}{2}\Phi V^2 T \tag{22}$$

$$= \frac{1}{2}m_p V^2 \tag{23}$$

A rocket converts the potential energy stored in its fuel into kinetic energy. The amount of energy produced is given by Einsteins famous mass-energy relation $E = mc^2$. This relation leads to the idea to consider the enigine as a mass burner. It converts mass into energy according to Einsteins mass-energy relation. Let Δm_p be the amount of fuel mass m_p converted into energy. The following amount of energy is produced

$$E_{produced} = \Delta m_p c^2 \tag{24}$$

For our ideal rocket this is all converted into kinetic energy, thus (using (23))

$$E_{produced} = \Delta E \tag{25}$$

$$\Rightarrow \Delta m_p c^2 = \frac{1}{2}m_p V^2 \tag{26}$$

$$\Rightarrow V = \sqrt{\frac{2\Delta m_p}{m_p}}\,c \tag{27}$$

Combining this with (13) gives

$$v_{end} = \sqrt{\frac{2\Delta m_p}{m_p}} \ln\left(1 + \frac{m_p}{m_e}\right) c \tag{28}$$

Thus for a direct propulsion rocket the exhaust and end velocity are given by (27) and (28). Remember this is true for an ideal rocket. Real life rockets will be less efficient. For real life rockets these equations impose the upper limits for those velocities.

The rocket we considered so far is an ideal one. But even for an ideal rocket not all potential energy contained in its fuel is converted in usefull kinetic energy. A lot of energy is lost in kinetic energy of the exhaust gases. The usefull effiency η is given by

$$\eta = \frac{E_{usefull}}{E_{produced}} \tag{29}$$

$$= \frac{\frac{1}{2} m_e v_{end}^2}{\frac{1}{2} m_p V^2} \tag{30}$$

$$= \frac{m_e \ln^2\left(1 + \frac{m_p}{m_e}\right)}{m_p} \to 0 \quad \text{for} \quad \frac{m_p}{m_e} \to \infty \tag{31}$$

So even for an ideal rocket the usefull efficiency approaches zero when it takes along a lot of propellant.

4 Indirect versus Direct Propulsion Rockets

So far we have only considered direct propelled rockets, ie. fuel and propulsion substance are the same. Now we would like consider indirectly propelled rockets. ie. fuel and propulsion substance are two different substances. An illustration of such a rocket is given in figure (2).

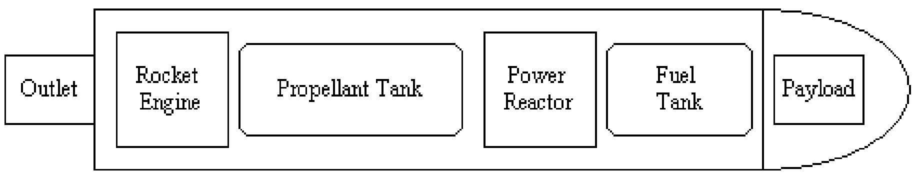

Figure 2: Indirect propulsion rocket

The equations describing this type of rocket are similar, except for one. Replace in all formulae m_e with m_{ei}, which is the mass of an empty (of propellant) indirectly propelled rocket. The only equation which differs significantly is (24), which now of course becomes (with m_f = fuel mass)

$$E_{produced} = \Delta m_f c^2 \tag{32}$$

Combining this with (23) leads to (with V_i = exhaust velocity of indirectly propelled rocket)

$$\frac{1}{2} m_p V_i^2 = \Delta m_f c^2 \tag{33}$$

$$\Rightarrow V_i^2 = \frac{2\Delta m_f}{m_p} c^2 \tag{34}$$

$$= \frac{2\Delta m_f}{m_f} \frac{m_f}{m_p} c^2 \tag{35}$$

Combining this with (27) gives (with V_d = exhaust velocity of directly propelled rocket)

$$V_i = \sqrt{\frac{m_f}{m_p}}\, V_d \tag{36}$$

We will now compare a directly and an indirectly propelled rocket. For comparison purposes both rockets have the same usefull carge (payload) and the same amount of propellant mass. The following relations are valid for both rockets

$$m_{ed} = m_{payload} + m_{body,d} \tag{37}$$

$$m_{ei} = m_{payload} + m_{body,i} + m_f \tag{38}$$

With m_{ed} and m_{ei} being respectively the mass of the empty of propellant direct and indirect rocket. And $m_{body,d}$ and $m_{body,i}$ their respective body masses. The body mass of a rocket is its mass when it has no payload, no propellant and no fuel on board. It is safe to asume that an indirect rocket will be heavyer than a direct one, which leads to

$$m_{body,i} > m_{body,d} \tag{39}$$

$$(37) \text{ and } (38) \text{ and } (39) \Rightarrow m_{ei} > m_{ed} \tag{40}$$

Combining this with (13) and using (36), (40), $\alpha_d = m_p/m_{ed}$ and $\alpha_i = m_p/m_{ei}$ gives

$$\frac{v_{end,i}}{v_{end,d}} = \frac{V_i}{V_d} \frac{\ln(1+\alpha_i)}{\ln(1+\alpha_d)} \tag{41}$$

$$= \sqrt{\frac{m_f}{m_p}} \frac{\ln(1+\alpha_i)}{\ln(1+\alpha_e)} \tag{42}$$

Now realising that

$$(40) \Rightarrow \alpha_i < \alpha_d \tag{43}$$

$$\Rightarrow \ln(1+\alpha_i)/\ln(1+\alpha_d) < 1 \tag{44}$$

This means that the only way for (42) to be greater than 1 is when $m_f > m_p$. Continuing with all this we find

$$
\begin{aligned}
(42) \Rightarrow \frac{v_{end,i}}{v_{end,d}} &< \sqrt{\frac{m_f}{m_p}}\frac{\alpha_i}{\ln(1+\alpha_d)} && (45)\\
&< \sqrt{\frac{m_f}{m_p}}\frac{m_p/m_f}{\ln(1+\alpha_d)} \qquad (\text{because } \alpha_i < m_p/m_f) && (46)\\
&= \sqrt{\frac{m_p}{m_f}}\frac{1}{\ln(1+\alpha_d)} && (47)\\
&< \frac{1}{\ln(1+\alpha_d)} \qquad (\text{because we took } m_f > m_p) && (48)\\
&< 1 \qquad \text{for } \alpha_d > e-1 \approx 1.7 && (49)
\end{aligned}
$$

Concluding, for high values of α_d, an indirect rocket is always slower than a direct rocket!

5 Nuclear fusion rockets

Probably the most powerfull rocket we could ever hope to build is a nuclear fusion powered rocket. Assuming all velocities concerned are less than or approximately equal to $0.3c$ (with c = speed of light), relativistic effects can be ignored and the above deduced formulae are valid. Specifically, this means that the following ineqaulities should be true

$$
\begin{aligned}
V &< 0.3c \quad \text{or} \quad V \approx 0.3c && (50)\\
v_{end} &< 0.3c \quad \text{or} \quad v_{end} \approx 0.3c && (51)
\end{aligned}
$$

The speed of $0.3c$ is chosen because relativistic deviations at velocity v are generally proportional to $\gamma = \sqrt{1 - v^2/c^2}$. For $v = 0.3c$ we get $\gamma = 0.954$, meaning the deviations are less than 5%. Accurately enough for this analysis. Consider the following fusion reaction of two deuterion nuclei

$$
{}^2_1H + {}^2_1H \rightarrow {}^4_2He \qquad (52)
$$

The mass conversion ratio, or mass defect, for this reaction is: 0.0064 gram of energy is produced per gram of reactant. Or in other words

$$
\Delta m_f/m_f = 0.0064 \qquad (53)
$$

For direct propulsion rockets this leads to (and remember indirect propulsion rockets are slower)

$$
V_d = 0.113c \qquad (54)
$$

Thus (50) has been satisfied. Now substituting this into (13) we find

$$
\begin{aligned}
v_{end} &= 0.113c\ln(1+\alpha_d) && (55)\\
&< 0.3c \qquad \text{for } \alpha_d < 13.2 && (56)
\end{aligned}
$$

So equation (51) is satisfied for not too large values of α_d as well.

Nuclear fusion rockets will almost certainly be indirect propulsion rockets. The rocket will carry a nuclear fusion reactor which powers the rocket engine. An *Optimistic* estimate for the mass ratios for the rockets considered is

$$m_{payload} : m_{body,d} : m_{body,i} = 1 : 2 : 4 \quad (57)$$

$$\text{and} \qquad m_{ed} : m_f : m_p = 1 : 1 : 10 \quad (58)$$

This results in a end velocity $v_{end,i}$ for an indirect rocket of

$$\begin{aligned} v_{end,i} &= V_i \ln(1+\alpha_i) && (59) \\ &= \sqrt{\frac{m_f}{m_p}} \sqrt{\frac{2\Delta m_f}{m_f}} \ln(1+\alpha_i)\, c && (60) \\ &= \sqrt{\frac{1}{10}} \sqrt{2 \cdot 0.0064} \ln(1 + 10/(1+4+1))\, c && (61) \\ &= 0.035c && (62) \end{aligned}$$

6 Conclusion

The closed star to Earth, Proxima Centauri, is 4.2 lightyears away. A one way voyage to it at a speed of $0.035c$ (taken from (62)) would take 120 years. There are 40 stars within 16 lightyears of the Earth. But reaching any of those would take even longer. From this one can conclude we will never reach the stars using an indirect fusion powered rocket, because even a one way voyage would take to long. At best we could hope to build an unmanned space probe which reaches one of these nearby stars within a few hundred years.

An indirect rocket powered by nuclear fusion is most likely the fastest space vessel we will ever be able to build. Alternative approaches for space travel, such as anti-mater engines or reactors, worm holes and solar wind sailing, are much less feasable. In fact, most likely they are not feasable at all.

This leads to the conclusion that reaching the stars will never be possible for humans at all. At most they can be reached by unmanned probes. There is one positive side to this however. Aliens will have the same problems in reaching us. Unless the restrictions of: one way journeys only, and travel times of hundreds of years, are of no problem to them, they will never visit us. Nor have ever visited us. So rest asured, 'the Grey' will not pay you a visit tonight ...